Problem 1. Consider a stock in a binary one-period model with value 100 USD at time $t = 0$. We assume that the value increases by 7 percent in the up-state and decreases with 3 percent in the down state at $t = 1$. The risk free interest rate r is 5 percent.

(i) Calculate the risk neutral probability distribution.

(ii) Calculate the arbitrage free price c of a European call option written on this stock with strike price $K = 67$ USD.

Answer:

The risk neutral probabilities are $(q, 1 - q) = (0.8, 0.2)$ and the call price

$$c = \frac{38}{1.05} \simeq 36.2.$$

Problem 2. Patients arrive independently and at random to a medical facility with a mean of 5 patients per hour.

(i) What is the distribution of the number of patients arriving during a period of two hours?

(ii) What is the probability that exactly k patients arrive during a period of two hours for each k?

(iii) Let Y_1 denote the stochastic variable that measures the time from the opening of the clinic until the first (or more) patients arrive. What is the distribution of Y_1.

Answer:

(i) The Poisson distribution with parameter $\lambda = 10$.

(ii) Let X denote the stochastic variable that counts the number of patients in a two hour period. Then

$$P[X = k] = e^{-\lambda} \frac{\lambda^k}{k!} \,,$$

where $\lambda = 10$.

(iii) Y_1 is exponentially distributed with parameter $\mu = 5$.

Problem 3. Let B_t denote the Brownian motion. Show that the expectation

$$E\big[\exp(B_t)\big] = e^{t/2} \qquad t \geq 0.$$

Answer:

$\exp(B_t)$ is a special example of the geometric Brownian motion

$$X_t = \exp(\mu t + \sigma B_t),$$

which is known to have mean $E[X_t] = \exp\big(\mu + \sigma^2/2\big)t$. Setting $\mu = 0$ and $\sigma = 1$ we obtain $E\big[\exp(B_t)\big] = e^{t/2}$.

Problem 4. Let B_t denote the Brownian motion and take $c > 0$. Show that the process

$$\hat{B}_t = \frac{1}{c} B_{c^2 t} \qquad t \geq 0$$

also is a Brownian motion.

Answer:

It is clear that $\hat{B}_0 = 0$, and that $\hat{B}_t$ has stationary and independent increments. We also note that $\hat{B}_t$ is normally distributed with mean zero and variance

$$\operatorname{Var} \hat{B}_t = \frac{1}{c^2} \operatorname{Var} B_{c^2 t} = t.$$

Problem 5. We consider the Heath-Jarrow-Morton (HJM) framework for interest rate models. The forward rates $F(t,T)$ are modeled by specifying the dynamics

$$dF(t,T) = \alpha(t,T)\, dt + \sigma(t,T)\, dB_t\,.$$

Assume that the forward rate volatility $\sigma(t,T)$ is given by

$$\sigma(t,T) = \sigma e^{-(T-t)} \qquad 0 \leq t \leq T,$$

where $\sigma > 0$ is a constant.

(i) Calculate the forward rate drift $\alpha(t,T)$ for $0 \leq t \leq T$.

(ii) Show that $\alpha(t,T) \to 0$ for $t \to T$.

Answer:

(i) We use the drift condition in the HJM model and obtain

$$\alpha(t,T) = \sigma^2 e^{-(T-t)}\big(1 - e^{-(T-t)}\big).$$

(ii) It follows that $\alpha(t,T) \to 0$ for $t \to T$.